EXERCISE SHEET 1

1) In this exercise I want to grumpily claim that the name "analytic class number formula" is a bit of a misnomer, in the sense that it is not so easy to use it as a means for computing class numbers.

   (a) Use the analytic class number formula to find the residue of $\zeta_{\mathbb{Q}(i)}(s)$ at $s = 1$.
   (b) Now compute the residue of $\zeta_{\mathbb{Q}(i)}(s)$ at $s = 1$ by other means, without using that $\mathbb{Z}[i]$ is a PID, so that you can use the analytic class number formula to actually compute the class number of $\mathbb{Q}(i)$. (I don't know how to do this! Update: I do know how to do this now, because Rob Rockwood told me how. Thanks Rob!)

2) Let X be an irreducible algebraic variety over a field k. Write $Z^n(X) := Z^n(X, 0)$ for the codimension n cycles on X. We say that a cycle $Z \in Z^n(X)$ is *rationally equivalent to zero* if there is a cycle $T \in Z^n(X \times \mathbb{P}^1_k)$ which projects dominantly onto $\mathbb{P}^1_k$ and two points $a, b \in \mathbb{P}^1_k$ such that

$$Z = T(b) - T(a)$$

   where $T(t) := (\mathrm{pr}_{X*}(T \cdot (t \times X))$ for a point $t \in \mathbb{P}^1_k$. In other words, "Z can be deformed to 0 along a $\mathbb{P}^1$". We say that $Z_1, Z_2 \in Z^n(X)$ are *rationally equivalent*, and write $Z_1 \sim_{\mathrm{rat}} Z_2$, if $Z_1 - Z_2$ is rationally equivalent to zero. It is easy to check that $\sim_{\mathrm{rat}}$ forms an equivalence relation. The *Chow group* of codimension n cycles on X is defined to be

$$\mathrm{CH}^n(X) := Z^n(X)/\sim_{\mathrm{rat}} .$$

   It turns out that we can replace $\mathbb{P}^1_k$ by any rational variety in the definition of "rationally equivalent to zero" (or even unirational variety). Show that $\mathrm{CH}^n(X) \cong \mathrm{CH}^n(X, 0)$.

3) Show that $\mathrm{CH}^i(X, n) = 0$ for $i > n + \dim X$.

4) Let L/K be a finite extension of fields. Consider the pullback map

$$\mathrm{CH}^i(K, n) \to \mathrm{CH}^i(L, n) .$$

   (a) Show that the pullback map is injective after tensoring with $\mathbb{Q}$. (This fact generalises to any finite flat map of varieties).
   (b) Give an example where the pullback map is not injective integrally. (The following theorem of Nesterenko-Suslin-Totaro might be useful: for any field F, $\mathrm{CH}^n(F, n) \cong K^{\mathrm{Mil}}_n(F)$ where the latter denotes the n-th Milnor K-group of F.)

   (c) Let k be a field. Compute the motivic cohomology $H^i_{\mathcal{M}}(\mathbb{P}^m_k, \mathbb{Z}(n))$ of projective m-space $\mathbb{P}^m_k$ in terms of the motivic cohomology of $\operatorname{Spec} k$.